Probability, An Introduction

PROBABILITY IS AS INTIMATELY RELATED TO COUNTING PROCESSES AS YOU ARE TO YOUR SKELETON! WITHOUT COUNTING WE CANNOT COVER THE CONCEPT EXCEPT IN FUZZY GENERALITIES. BY THE END OF THIS SECTION YOU SHOULD BELIEVE THIS.

However, we do need to recognize some differences between our counting vocabulary and the probability vocabulary. Let’s begin with a definition of probability.

Probability is the study of the likelihood or chance that an outcome or event can occur.

This sentence is replete with words that need explanation!

An ***outcome*** is some basic result that we can record in a process.

Example: Say we flip a coin. The ***outcomes*** are to see *heads* or *tails*. We discount the *on-edge* possibility as a nonevent.

Example: We spin a dial with areas marked *a, b, c*. Then the outcomes are *a*, *b*, or *c*.

A ***sample space***, usually denoted by *S*, is the collection of all possible ***outcomes*** from some action or process.

Example: Say we flip a coin again. The outcomes are still *heads* or a *tails*. These are the only two possible outcomes. We discount “edgies” and require a re-toss just as in most sporting events. The sample space is:

S = {heads, tails}

Example: Say we flip a coin two times. The outcomes are of each spin are *heads* or *tails*. However, the sample space is composed of the results of both spins. We list them as ordered pairs.

S = {(heads, heads), (heads, tails), (tails, heads) (tails, tails)}

The number of elements in the sample space is simply an application of the multiplication principle. There are two choices at each flip. So there are $2 \times 2 = 2^2 = 4$ outcomes.

Example: We spin a dial with areas marked *a, b, c*.

S = {a, b, c}.

Example: We spin a dial with areas marked *a, b, c* three times. Similarly to the coin flip the outcomes are not *a*, *b*, or *c*. They are the ordered triples listed as (*first spin, second spin, third spin*). The sample space is listed in the table below.

aaa	aab	aac	aba	abb	abc	aca	acb	acc
baa	bab	bac	bba	bbb	bbc	bca	bcb	bcc
caa	cab	cac	cba	cbb	cbc	cca	ccb	ccc

Listing the 27 outcomes is tedious. Fortunately, we are more interested than the **count** of the set than the list itself in many cases. Again counting the sample space is an application of the multiplication principle.

There are three choices at each spin. The sample space has $3 \times 3 \times 3 = 3^3 = 27$ outcomes.

A word of caution about sample spaces: These are not universal sets, strictly speaking. Sets are *unordered*, complete listing of elements with each one listed *exactly once*. As we will see later, sometimes it is convenient to allow repetition in a ***sample space*** to better understand the probabilities.

An ***event*** is any combination of ***outcomes*** using words like *and, or, not, excluding*, etc.

Likelihood of an event implies that there should be some quantification or numerical value assigned to a probability. Words like *certain*, *possible*, or *impossible* are used to quantify likelihood. To make statements in a more succinct way, we adopt the notation that ***P***(*X*) stands for the *probability that situation X occurs*. We can make these numerical assignments without any other information:

- ***Impossible*** - event *X* simply cannot happen, so $P(X) = 0 = 0\%$.

  Example: We want the likelihood of *picking an apple from a pine tree*. The probability of that silly event is zero, zip, *nada*. It is impossible in any natural way. Call it *impossible*.

- ***Certain*** - event *X* must happen, so $P(X) = 1 = 100\%$

  Example: We want the likelihood of *finding an apple tree in an apple orchard*, the probability of that event is a certainty. We assign the value 100% = 1 unit. The most certain you can be is 100%.

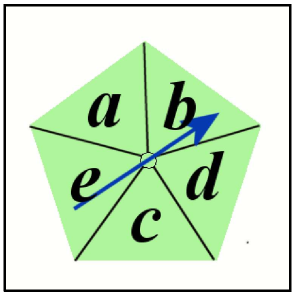
  Notice the wording. We did not say *"finding only apple trees in the orchard"* or even *"finding apples in an apple orchard."* These events are *not* certain. There may be other trees that have been allowed to remain in the orchard, but why call it an apple orchard if it doesn't have apple trees? Also, if it is too early in the growing season, there may be no apples present yet.

  > **Philosophy 101:** While it may be encouraging to have someone say they'll give 110%, it is impossible. You cannot give more than you have, period. You can, however, admit you've been holding out in your previous efforts! Shame on you! Then you might be able to increase your effort by 110%.

- ***Possible*** - event *X* can happen, so $0 < P(X) \le 1 \left(or, 0\% < P(X) \le 100\% \right)$. Whatever event *X* is, we should include it in the thoughts about likelihood. Notice that since the event is possible, $P(X) \ne 0$.

  Example: We want the likelihood that in a group of 364 people two of them have the same birthday. By birthday we mean month and day but disregard the year. Since there are 366 possible birthdays (don't forget leap years), we cannot state that this is a certainty. However, with only two days not covered, it is almost inconceivable that the group doesn't have two with matching birthdays.

Next let's use some simple examples to motivate a formula for probability. The easiest one with some sense of depth to it is a *spinner*. We want to play a game that requires us to spin a spinner one time at each of our turns. Suppose the spinner has five equally spaced divisions. One is shown to the right. Since the spinner is labeled with ***a, b, c, d,*** or ***e,*** the *sample space* is $S = \{a, b, c, d, e\}$.



This spinner is typical of one used in a child's game. All divisions are equal. So the chances of an ***a*** occurring are 1 in 5. We would write $P(a) = {}^{1}\!/_{5} = 0.2 = 20\%$.

Notice that the answer to a probability problem can be in a fractional form, decimal form, or as a percentage. The best form of answer is the one that registers best with the person you are providing the information to.

Since the divisions are equal, results ***a, b, c, d,*** and ***e*** each have the same probability of happening. We have a ***uniform*** chance of getting any of the letters in one spin.

The next step should be to compile all the probabilities in some usable form. All possible outcomes with the associated probabilities are called a ***probability model*****.** The table below would be the *model* for this spinner for one spin. Notice that all outcomes total to $1 = 100\%$. This must happen to have a correct probability model.

Outcome	***a***	***b***	***c***	***d***	***e***	***Total***
P(Outcome)	0.2	0.2	0.2	0.2	0.2	1.0

We can stop here and summarize the ground rules for a correctly constructed ***probability model***:

$$P(S) = 1 = 100\% \text{ and } 0 < P(outcome) < 1$$

The rules tell us that we should never have negative numbers or zero listed as probabilities, and that all probabilities must add to one. We can also write the simplest formula for calculating probabilities where counting alone will give us all we need.

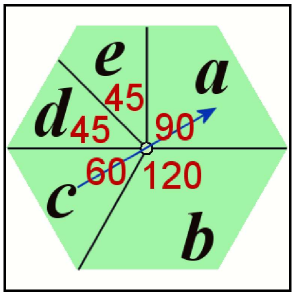
The probability that event X will occur is calculated as the <u>ratio</u> of ways our event can happen to the number of possible events.

$$P(X) = \frac{\text{count of ways } X \text{ happens}}{\text{count of all possible outcomes}} = \frac{c(X)}{c(S)}$$

Take a moment and calculate or recognize the following probabilities. After you have your answers, click on the check mark to the right. ✔ (or paste this link: http://fym.la.asu.edu/~mat142/Doug/Probability/Documents/Lesson_2.2.1_Problem_Solutions_1.PDF

1. $P(d)$
2. $P(a \text{ and} d)$
3. $P(a \text{ or } d)$
4. $P(vowel)$, where ***vowel*** = {***a, e, I, o, u***}
5. $P(consonant)$, where we define ***consonant*** as the ***complement*** of vowel.

Okay so you got them all right. Let’s see how good you really are. Suppose our spinner looks like this.



The probability model is going to look very different. First, this is *not* uniform. Different outcomes have different chances of occurring. The basis for our assigning probabilities is that a circle has 360°. So we can assign probabilities by using the *degrees, or weights,* provided in each division.

Conceptually, we could subdivide each section and end up with 90 ***a*** sectors, 120 ***b*** sectors, etc. So our table looks like the one below. Notice that we did not reduce the fractions. Since we will almost certainly add some of these values, we don’t want to lose that common denominator.

Outcome	***a***	***b***	***c***	***d***	***e***
P(Outcome)	$\frac{90}{360}$	$\frac{120}{360}$	$\frac{60}{360}$	$\frac{45}{360}$	$\frac{45}{360}$

Take another moment and calculate or recognize the following probabilities. After you have your answers, click on the check mark to the right. ✔(or paste this link http://fym.la.asu.edu/~mat142/Doug/Probability/Documents/Lesson_2.2.1_Problem_Solutions_2.PDF)

1. $P(d)$

2. $P(a \text{ and } d)$

3. $P(a \text{ or } d)$

4. $P(vowel)$, where ***vowel*** = {***a, e, I, o, u***}

5. $P(consonant)$, where we define ***consonant*** as the ***complement*** of vowel.

We’re in the home stretch! All of the properties of probability that are fit to print are within reach.

The following should be reasonably clear from what we did in the examples:

But what about situations where events don’t exclude each other? We fall back to a formula used in counting to create a formula for probability.

Suppose for that last spinner, we are asked to calculate the probability that we would spin (one time only) to get either a *vowel* or a letter in the word "*bead*."

We already know that the $P(vowel) = P(a) + P(e) = \frac{90}{360} + \frac{45}{360} = \frac{135}{360}$.

We can calculate the probability for getting a letter in the word "*bead*" as

$$\boxed{P(bead) = P(b \text{ or } e \text{ or } a \text{ or } d) = P(b) + P(a) + P(e) + P(d) = \frac{120+45+90+45}{360} = \frac{300}{360}}$$

Some of you have already realized that the only letter not used in "*bead*" from the spinner is ***c***. It would have been quicker to calculate $P(bead) = 1 - P(c) = 1 - \frac{60}{360} = \frac{300}{360}$.

However, $P(vowel \text{ or } bead) = P(vowel) + P(bead) = \frac{135}{360} + \frac{300}{360} = \frac{435}{360} > 1$!!!

This cannot happen. You have probably spotted the problem. Two vowels are included in "*bead.*" We have an overlap in the sets of letters. We must also have an overlap in the calculations for the probability.

$$\boxed{c(X \text{ or } Y) = c(X) + c(Y) - c(X \cap Y)}$$

$$\frac{c(X \text{ or } Y)}{c(S)} = \frac{c(X) + c(Y) - c(X \cap Y)}{c(S)}$$

$$\frac{c(X \text{ or } Y)}{c(S)} = \frac{c(X)}{c(S)} + \frac{c(Y)}{c(S)} - \frac{c(X \cap Y)}{c(S)}$$

$$\boxed{P(X \text{ or } Y) = P(X) + P(Y) - P(X \cap Y)}$$

Recall, that we had a counting formula for handling a two-set overlap. In the part that follows, we quickly convert it into a formula for probabilities when two events, call them ***X*** and ***Y***, overlap. So, $X \cap Y \neq \varnothing$.

Nothing in this is mysterious.

We just used basic facts about fractions to rewrite the counting formula in terms then recognize that each term is a probability statement.

The bottom line formula, and all of the process to get it, comes into play regularly as we calculate probabilities. Get very familiar with this formula.

$P(S) = 1$	Your "event" is to choose from the sample space. Cannot miss!
$P(\varnothing) = 0$	Your trying to make an impossible choice!
$0 < P(X) < 1$	Your event is possible, but not certain.
$P(\overline{X}) = 1 - P(X)$	Complementary events must include all of S.
$P(X \text{ or } Y) = P(X) + P(Y)$	For events that exclude each other, just add the probabilities .

Okay! The story is complete.

The basics of probability require exactly the same skills in counting that you used earlier in this course. ***Counting is the skeleton for probability***. But just as with the average living *invertebrate*, the skeleton is sometimes not so visible. In later problems we'll develop methods for answering questions where counting is not all that we need.

Example

Suppose a committee is made up of 8 students, 12 teachers and 10 parents with no person being in more than one group. Let us use S for students, T for teachers and P for parents. This means that the committee is made up of 30 people. Suppose that we want to select (draw) one person from the committee.

1. What is the probability that we select a student?
2. What is the probability that we select a parent?
3. What is the probability that we select a student and a teacher (in one draw)?
4. What is the probability that we select a student or a teacher (in one draw)?

Solution

1. $P(S) = \frac{8}{30} = \frac{4}{15}$
2. $P(P) = \frac{10}{30} = \frac{1}{3}$
3. Impossible! $P(S \text{ and } T) = \frac{0}{30} = 0$
4. $P(S \text{ or } T) = \frac{8}{30} + \frac{12}{30} = \frac{20}{30} = \frac{2}{3}$

Suppose 4 of the teachers are also parents. This means that the committee is now 26, not 30. Carrying two titles does not mean they become two people!

1. What is the probability that we select a parent that is also a teacher?
2. What is the probability that we select a parent or a teacher?

Solution

1. $P(P \text{ and } T) = \frac{4}{26} = \frac{2}{13}$
2. $P(S \text{ or } T) = P(P) + P(T) - P(S \text{ and } T) = \frac{10}{26} + \frac{12}{26} - \frac{4}{26} = \frac{18}{26} = \frac{9}{13}$

Example: You are allowed to draw from a bag containing 50 equally sized balls while blind-folded. There are 15 white balls, 10 black balls, 6 red ball, one gold ball and the remaining balls of are of other undisclosed colors.

1. What is the probability that you would choose the gold ball in a blind drawing? $P(gold) = \frac{1}{50}$
2. What it sh probability of not drawing a gold ball? $P(not\ gold) = 1 - \frac{1}{50} = \frac{49}{50}$
3. What is the probability of not drawing a red ball? $P(not\ red) = \frac{50-6}{50} = \frac{44}{50}$
4. What is the probability of drawing *one* of the undisclosed colors? "Strictly between zero and 100%." is about all you can say. We do not know if there are 18 distinct undisclosed colors so any stronger statement is speculative.
5. What is the probability of not drawing any undisclosed color? $P(not\ undisclosed) = 1 - \frac{18}{50} = \frac{32}{50}$

   We can answer this since all undisclosed colors are eliminated.