Objectives: At the end of this lesson, you should be able to:
1. Create a matrix from a linear system
2. Identify the various kinds of matrices we can create from them.
3. Define Gaussian Elimination within matrices
4. Apply Gaussian Elimination to a matrix.

Background

Remember synthetic division? It made polynomial division simpler by relying only on the position of the various degrees of the variable.

It’s not an uncommon practice in math to let position dictate size, degree or otherwise relate to a number such as a coefficient. We do it with numbers all the time. That’s how we know that 10001 is a lot more than 101.

We do exactly the same thing with linear systems. Think back to all of our Gaussian Conventions. We always said we would add or otherwise combine the columns of like variables. If we just drop out the variables, we create an array of coefficients or constants.

As long as we don’t jumble the order on any row, position alone should allow us to do the various row operations.

You may have already done this in the set-up for linear optimization. However, it won’t hurt to review the process.

**Gaussian Elimination in Matrices**

Take a look at this system:

\[
\begin{align*}
\begin{pmatrix}
\mathbf{r}_1 : & x_1 - 3x_2 + 3x_3 = 5 \\
\mathbf{r}_2 : & 2x_1 - 4x_2 + 2x_3 = 1 \\
\mathbf{r}_3 : & 4x_1 - 5x_2 + 2x_3 = 2 
\end{pmatrix}
\]

Rectangular arrangements of numbers are called matrices. (Matrix is the singular). We can create a number of arrays from this system. Suppose we decide to just look at the coefficients. The system creates all of these matrices:

<table>
<thead>
<tr>
<th>Coefficient Matrix</th>
<th>Variable Matrix</th>
<th>Constant Matrix</th>
<th>Augmented Matrix</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
1 & 3 & 3 \\
2 & 4 & 2 \\
4 & 5 & 2 
\end{pmatrix}
\] | \[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 
\end{pmatrix}
\] | \[
\begin{pmatrix}
5 \\
1 \\
2 
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 3 & 3 & 5 \\
2 & 4 & 2 & 1 \\
4 & 5 & 2 & 2 
\end{pmatrix}
\] |

1Gaussian Elimination is also called Gauss-Jordan Reduction. Either term refers to using the standard row operations to find the rref form of the matrix.
Matrices

We would create the coefficient matrix related to the system. We could create the matrix of only the variables, aptly named the variable matrix. Or, we could create the constant matrix. Each of these has use in working with systems and the related matrix forms.

The last special form is called the augmented matrix. This is the variable matrix augmented by the constant matrix. By the position of values we can relate back to the variable. An augmented matrix is identifiable because we usually place a dotted vertical line where the equal sign would be. Here’s some more good news. You already know everything useful about working with an augmented matrix. We would transform this matrix to a solution matrix by exactly the same conventions we used in the system.

### Using the Augmented Matrix

Let’s solve this system through the augmented matrix. I choose to eliminate (zero out) the entries in the first column below the first row.

By observation, I decided to use the linear combinations below.

\[
\begin{bmatrix}
1 & 3 & 3 & 5 \\
2 & 4 & 2 & 1 \\
4 & 5 & 2 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
R_2 = 2r_1 - r_2 \\
R_3 = 4r_1 - r_3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 3 & 5 \\
0 & 2 & 4 & 9 \\
0 & 7 & 10 & 18
\end{bmatrix}
\]

So far so good, we have a nice process going. Notice also that I did two steps at once. Since I wasn’t going to interact rows 2 and 3, it wasn’t that hard to manipulate both at once.

Now again by observation I see that I can eliminate the \(a_{32}\) entry using the linear combination below.

\[
\rightarrow R_3 = 7r_2 - 2r_3 \rightarrow
\begin{bmatrix}
1 & 3 & 3 & 5 \\
0 & 2 & 4 & 9 \\
0 & 0 & 8 & 27
\end{bmatrix}
\]\n
Finally, to make my life easier, I’ll scale each row to get the staircase of one’s on the diagonal.

\[
\rightarrow R_2 = \frac{1}{2} r_2 \\
\rightarrow R_3 = \frac{1}{8} r_3
\rightarrow
\begin{bmatrix}
1 & 3 & 3 & 5 \\
0 & 1 & 2 & \frac{9}{2} \\
0 & 0 & 1 & \frac{27}{8}
\end{bmatrix}
\]

You should recognize this as the row echelon form. From here we could back-substitute to reach the solution. However, I want to take this to the reduced row echelon form. From that form we can read the solution to the original system. Let’s start from the bottom since it has zeros in all but the last position.

\[
\rightarrow R_2 = 2r_3 - r_2 \rightarrow
\begin{bmatrix}
1 & 3 & 3 & 5 \\
0 & 1 & 0 & -\frac{9}{4} \\
0 & 0 & 1 & \frac{27}{8}
\end{bmatrix}
\]

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2  Sadly, the homework doesn’t abide by our little “r” big “R” convention. However, it doesn’t prevent you from doing what the problems ask.
Matrices

At the risk of burning out an already stressed brain, I going to zero out both $a_{12}$ and $a_{13}$. I can do this because both elements have a 3 value. Otherwise this might take a couple of steps. Notice the linear combination uses all three rows. Neat!

$$\rightarrow R_2 = r_1 - 3(r_3 + r_2) \rightarrow \begin{bmatrix} 1 & 0 & 0 & 13/8 \\ 0 & 1 & 0 & -9/4 \\ 0 & 0 & 1 & 27/8 \end{bmatrix}$$

The solution is now readable as the ordered triple $\left(\frac{13}{8}, \frac{-9}{4}, \frac{27}{8}\right)$.

Here’s where we get into the wonderful world of answer display.

This solution can also be reflected as the column matrix to the right. (A column matrix has a single column.) However, the appropriate way to interpret this is by realizing that it is supposed to be juxtaposed with the variable matrix as shown.

Also notice that I did revert to an exact decimal representation. The arithmetic in this problem could have been done totally in decimal. However, had we divisors of 3, 7, 11 or other irritating denominators that create repeating decimals, the work must be done using rational fractions. Bad things happen otherwise.

Using Gaussian Elimination processes within an augmented matrix is modestly simpler than using the original system since we don’t need to reproduce all the variables. However, this is still a time consuming process for large systems, even with only 4 or 5 variables. In the next lesson we will use the other matrix forms, coefficient, variable and constant, to develop better methodologies for large systems.