Multiplication Principle

Background

You have looked at counting processes involving small sets. In most cases, if necessary, we could go element by element to see how many elements fit into various characteristics. However, we regularly take a small number of elements and jumble them, or arrange them, to create an extremely large set of objects.

Take the digits \{0, 1, 2, 3, \ldots, 9\}. By using them in different arrangements we create all of the infinitely many whole numbers. Throw in a decimal point or a minus sign and the infinitely large set of possibilities gets even larger!

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We have one essential concept, called the multiplication principle. This is a simple thought process. Let's illustrate it with two small sets.

The set \textit{cat} = \{c, a, t\} has three elements. The set \textit{dogs} = \{d, o, g, s\} has four. Suppose we decide to choose one letter from each set, first from \{c, a, t\} then from \{d, o, g, s\}. How many ways can we choose a two-letter arrangement if we are not allowed to reuse any letter within either set?

We usually call these arrangements \textit{words}. However, the arrangements do not need to be real words. Then we append the phrase \textit{real or fictitious} to emphasize that fact. We also sometimes refer to them as “\textit{strings}” of letters. The main thing is that words are \textit{ordered arrangements}.

It's easy to see that we have three ways to choose from the first set and four from the second. The multiplication principle tells us we have \(3 \times 4 = 12\) possible arrangements.

Another way to visualize this is through a \textit{tree diagram}. In a tree we show the step by step possibilities in choosing. The \textit{tree} for this situation looks like the one to the left. Count the branches at the right. There are 12 of them. The fruit of the tree is the two-letter combinations \{cd, co, cg, cs, ad, ao, ag, as, td, to, tg, ts\}.

For combining small sets, \textit{trees} are reasonable for listing and counting results. Try it for the letters in the words \textit{warm} and \textit{night}. There are \(4 \times 5 = 20\) arrangements. Now try it for \textit{cauliflower} and \textit{healthy}. There are \(12 \times 7 = 84\) arrangements. That will take a large sheet of paper.

As another example of the power of the multiplication principle, suppose we could choose from the letters in the three words \textit{you}, \textit{need}, and \textit{math}. You choose one letter from each word’s set of letters in order. How many three-letter arrangements are possible?

We apply the multiplication principle \textit{wisely}. We have 3 choices from \textit{you}, 4 from \textit{need} and 4 choices from \textit{math} . . . Here's where the wisdom comes in, we have only 3 distinct choices from \textit{need}. Even though it is a 4-letter word, we have repeated letters. As we will see later in probability, the repetitions affect our likelihood of choosing an \textit{e} over \textit{d} or \textit{n}. The number of arrangements discounts duplications in choosing. Just as a preview of the next section on probability the word \textit{yet} will appear in our listing two times since there are two \textit{e}'s in \textit{need}. It is twice as likely to be spelled in our drawing process.
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However, in counting the number of arrangements we must eliminate the duplications since the list of all possible arrangements must follow our rules for sets. The set created by need is \{n, e, d\}. So by the multiplication principle we have \(3 \times 3 \times 4 = 36\) possible distinct arrangements.

Permutations

The next step is to work on situations where we draw sequentially from the same set. There are two variations on this theme: We can draw only to get a number of elements, or we can concern ourselves with the order of the elements we draw.

When we choose from the same set without replacing our choices and observe the order of the elements we have drawn, we have created a permutation.

Example: Create all three-letter strings from the letters in cat.

The multiplication principle says, that we have 3 ways to choose the first letter, 2 ways for the second and only 1 for the last. At that point we have exhausted the letters, so the process stops.

The number of arrangements where we use all letters in the order selected is \(3 \times 2 \times 1 = 6\). Much more simply stated: There are six permutations. The list is simple: act, atc, cat, cta, tac, tca.

There! We've followed the instructions and created them. We also know that there should be only six, so we can stop looking for other possibilities.

Example: Create all permutations with 26-letter strings using the standard English alphabet!

You do it. I refuse! However, if you want me to count the possible permutations, that is easy. It goes this way: \(26 \times 25 \times 24 \times 23 \times 22 \times 21 \times \ldots \times 2 \times 1 = 4.032914611 \times 10^{26}\)

This count is so large that my calculator cannot display the exact value. We should have a 27-digit number. Here's where notation comes in handy. Any multiplication sequence that counts down from a natural number to end with times 1 can be represented by the ! symbol. This is called factorial notation.

So for the last two examples, the first result is \(3! = 3 \times 2 \times 1 = 6\) and the second is \(26 \times 25 \times 24 \times \ldots \times 2 \times 1 = 26!\)

In general, counting down from \(n\) to 1 and multiplying as we go is

\[n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1\]

The next step is to choose only a few letters. Suppose we wanted to get only the 4-letter strings where order matter. Again we plan to choose and keep, so this is typically called "without replacement." This is more simply described as the 4-letter permutations chosen from the alphabet.

Procedurally, this is easier than our previous example. There are only \(26 \times 25 \times 24 \times 23 = 358800\) permutations!

But suppose we wanted to choose 10-letter permutations. It would be very tedious to type in 10 factors. Well, we have a formula. It is easy to see where it comes from because it just depends on rewriting fractions.
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Start with the factorial definition: \( n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 \)

Now know we only want the first \( k \) of them. This is the technical step. I've bracketed the first \( k \) factors from the left to get \( n! = \left[ n \times (n-1) \times (n-2) \times \ldots \times (n-k) \right] \times (n-k) \times \ldots \times 3 \times 2 \times 1 \)

So all the stuff after (outside) the brackets is extraneous to our counting process. So we just divide it away!

\[
\frac{n!}{(n-k) \times \ldots \times 3 \times 2 \times 1} = \frac{\left[ n \times (n-1) \times (n-2) \times \ldots \times (n-k-1) \right] \times (n-k) \times \ldots \times 3 \times 2 \times 1}{(n-k) \times \ldots \times 3 \times 2 \times 1}
\]

\[
\frac{n!}{(n-k) \times \ldots \times 3 \times 2 \times 1} = \left[ n \times (n-1) \times (n-2) \times \ldots \times (n-k-1) \right]
\]

Now we do a little recognition. The factors \( (n-k) \times \ldots \times 3 \times 2 \times 1 \) \( (n-k)! \) are exactly

By substitution we get this result:

\[
\frac{n!}{(n-k)!} = \left[ n \times (n-1) \times (n-2) \times \ldots \times (n-k-1) \right]
\]

So for our 10-letter permutation process, we have

\[
\frac{26!}{(26-10)!} = 1.927522397 \times 10^{13}
\]

Somewhere on your calculator (look for it), you may see one of these symbols:

\( nP_k \quad nP_r \quad P(n, k) \quad P(n, r) \)

These symbols are used to calculate the number of permutations when we limit our choice to some \( k \) or \( r \) of the total of \( n \). The \( P \) stands for Permutation.

All of them represent the calculation \( \frac{n!}{(n-k)!} \).

**I will probably use \( P(n, k) \) as the abbreviation in other lessons. You should be able to recognize any of them.**

**Example:** How many 4-digit codes can be created from this set \{a, b, c, %, &, #, x, y, z\} if each character can only be used once?

The set has 9 distinct elements. We want to use only 4 of them. So we disregard 5 of them.

The calculation is \( P(9, 4) = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 3024 \). Simple.
Combinations

Our next step is to disregard order. If we choose sequentially and keep our choice at each step, we create a \textit{combination} of the symbols.

Just to emphasize the vocabulary –

- \textit{Arrangement} is a general term and denotes some method of choosing.
- \textit{Permutation} is a specific method of arranging where we choose sequentially, without replacement, and concern ourselves with the order of the objects chosen.
- \textit{Combinations} are arrangements where we choose sequentially, without replacement, but \textit{do not} concern ourselves with the order of the objects chosen.

Logically, there must be fewer combinations than permutations when we choose the same number \( k \) from a set with \( n \) objects. Let's use a small set again to motivate the way this works.

Count all 3-letter arrangements from the set \{b, e, d\} where we do not repeat letters and order does not matter.

Just as before if we listed all the \textit{permutations}, we would get the set \{bed, bde, ebd, edb, dbe, deb\}.

But if order does not matter, these are the same selection! There is only one (1) choice, to take them all.

Think of it as grabbing all three at one time.

Suppose we chose to take only 2 letters from the three in \textit{combination}. (Again, to beat the drum, \textit{combination} in this context has a very specific meaning.)

We can list all \textit{permutations} to get the set \{be, bd, eb, ed, de, db\}.

But if order doesn't matter, these pairings are equivalent \( be = eb, bd = db, ed = de \).

So we actually have only three possible \textit{combinations}. Thinking of it as grabbing two at a time.

Flip your thinking. If we took two, we must have left one of the set behind. There are exactly three ways to leave one element behind when you take two out of three without regard to order.

One more example, then we generalize.

Count all 3-letter arrangements from the set \{b, e, a, d\} where we do not repeat letters and order does not matter.

Hmmmm! If I flip my thinking, I must leave one behind at the end of the choosing process. There must only be four real \textit{combinations}. All done!

However, let's see what we see through the formulaic approach.

Listing all the possibilities of permutations (there are 24) is too tedious! Let's choose a single example and discuss it to death.

I choose to look at \textit{bea}. If I did list all of the 24 permutations, I would find these possibilities: \textit{bea, bae, eab, eba, bea, bae}. These six permutations constitute a single combination of \( b, e, \) and \( a \). They count as one.

Do that with any other three-letter combination. You will always find a 6-fold repetition of the combination.

So where does the \textit{six} come from? It is the number of permutations of the three letters we are looking at, specifically in \textit{bea}. 

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But, there are 3! ways to arrange any specific choice of three letters. Just as we adjusted the \( n! \) formula to get the permutation formula, we are now going to adjust the permutation formula to create a combination formula, by division. We divide by the 3! to adjust our count.

\[
\frac{P(4, 3)}{3!} = \frac{4!}{3!1!} = \frac{4!}{1!3!} = 4
\]

In this last example we have

\[
\frac{P(3, 3)}{3!} = \frac{3!}{3!0!} = \frac{3!}{0!3!} = 1
\]

This does require us to establish one significant fact: \( 0! = 1 \). Conceptually, it means that the only way to choose nothing is not to choose. That is the only (1) way to do it!

Somewhere on your calculator (look for it), you may see one of these symbols:

\( _nC_k \quad _nC_r \quad C(n, k) \quad C(n, r) \)

These symbols are used to calculate the number of combinations when we limit our choice to some \( k \) or \( r \) of the total of \( n \). The \( C \) stands for Combination.

All of them represent the calculation \( \frac{n!}{k!(n-k)!} \).

I will probably use \( C(n, k) \) as the abbreviation in other lessons. You should be able to recognize any of them.

We're into the home stretch. From this point it is a recognition process. We can mix the choosing process (arrangements, combinations, permutations) as needed.

Example: How many ways can we arrange the results of the following process?

1. Choose five-letter combinations from the set \{a, b, c, d, e, f\}.
2. Next, choose 3-number permutations from the set \{1, 2, 3, 4, 5\}.
3. Finally, select two different colors to paint your numbers and letters where one is the outline of the character and the other is the fill. You have 12 distinct colors of paint.

Here's the hard part. Recognize that this is a multiplication principle process between steps one, two and three. Whatever numbers come out of these steps is multiplied together.

Step 1 result: Easy! We have six letters, and we chose 5, so we left 1. There are 6 ways to leave one. Confirm this using the combination formula, \( C(6, 5) = 6 \).

Step 2 result: Easy! We have 5 distinct choices and need only three of them.

This is \( P(5, 3) = 60 \). Or by the multiplication principle, \( 5 \times 4 \times 3 = 60 \). Both are acceptable approaches.

Step 3 result: Easy! Well, maybe not as easy! Look at the two letters to the right. We would consider these to be different arrangements using the same two colors.
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So this third step is a *permutation* process. We would have 12 outline × 11 fill = 132 color arrangements because the outline must be *different* from the fill.

**Finally, there are** 6 × 60 × 132 = 47520 possible arrangements under this process.

Each step is relatively simple. The real work comes in following the instructions for each step. But suppose the last step said "You may select an outline and a fill color for the characters."

This is a much broader statement. It doesn't require the outline and fill to be different. So when we choose the fill, we have 12 choices. When we choose the outline, we still have 12 colors. So, we have 12 × 12 = 144 colors.

This is a *with replacement* process. Similarly, if we were allowed to choose at step 2 *with replacement*, we would have 5 × 5 × 5 = 5³ = 125 possible arrangements.

You may see a neat *with replacement* formula there.

\[
\text{If you choose } k \text{ times from } n \text{ objects with replacement where order matters,} \\
\text{there are } n^k \text{ possible arrangements.}
\]

So what if order doesn't matter, such as in step 1? It's back to the old adjustment routine. For each of the *k* objects you have *k!* duplications.

\[
\text{If you choose } k \text{ times from } n \text{ objects with replacement where order does not matter,} \\
you have \frac{n^k}{k!} \text{ distinct possible arrangements.}
\]

Here’s some food for thought about processes.

Spinners, coins, dice, dart boards, and targets in general are automatically *with replacement processes* since a choice or result is not removed as a possible result. (This assumes the target is not struck with a nuclear weapons or some equally destructive device that does tend to remove the target from consideration.)

Lotteries, card decks, committee selections, and election processes are usually without replacement processes since the choice or result of a step is removed from further consideration.

**Examples:**

1. Choose all possible 5-letter arrangements from the standard English alphabet if no letter is reused.
   
   This is just \( P(26, 5) \)

2. Choose all possible 5-letter arrangements from the standard English alphabet.
   
   This is \( 26^5 \) since no restriction on reuse is presented.

3. Create all possible arrangements from spinning a spinner with four distinct colors seven times.
   
   This is \( 4^7 \) since no restriction on reuse is presented.
4. Spell all words real or fictitious using all of the letters in the set \{c, o, m, b, n, e\} exactly once.

   This is just \( P(6, 6) = 6! \)

   By “picture,” we have six positions to fill: \( \underline{6} \underline{5} \underline{4} \underline{3} \underline{2} \underline{1} \) then multiply.

5. Spell all six-letter words, real or fictitious, using only the letters in the set \{c, o, m, b, n, e\}.

   This is \( 6^6 \) since no restriction on reuse is presented. Notice that famous word “eeeeee” is in this collection.

   By “picture,” we have six positions to fill: \( \underline{6} \underline{6} \underline{6} \underline{6} \underline{6} \underline{6} \) then multiply.

6. Spell all seven-letter words, real or fictitious, using all letters in the word "chicken" as often as they appear

   This begins as \( 7! \) since we are restricted to "as often as they appear." By “picture,” we have seven positions to fill: \( \underline{7} \underline{6} \underline{5} \underline{4} \underline{3} \underline{2} \underline{1} \). However the letter "e" is there 2 times. We need to adjust for that.

   So the results is \( \frac{7!}{2!} = \frac{5040}{2} = 2520 \).

As a final warning. There is an element of "blind luck" possible in these calculations. The result above is numerically the same as choosing 5 letters from a list of seven distinct letters, \( P(7,5) \). However, While the numbers agree, the logic does not. The logic is what counts. That is why we require you to show your work on these problems.

**On to Lesson 5**