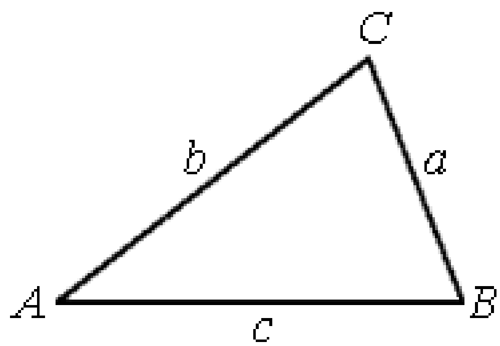
The Law of Tangents

Most geometry and pre-calculus books have entire units devoted to learning about two of the most important formulas in mathematics: the law of sines and the law of cosines. Both of these are very useful and very powerful tools, as they can be used to find unknown angles and sides of triangles. Yet, these books rarely devote anything to another formula that is equally powerful and useful: the law of tangents. The law of tangents can be used whenever two sides and one of the opposite angles are known. It can also be used when two angles and one of their opposite sides are known.

The law of tangents:

$$\frac{a+b}{a-b} = \frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$$



There are many different ways that the law of tangents may be derived. The following demonstrates one way this can be done.

First, let there be a circle with a radius of one unit (in geometry terms, this is called a *unit circle*) and let it be superimposed onto a Cartesian coordinate plane, where the center of the unit circle is at the origin. Let there also be two points defined on the circle, called A and B, where point A $\neq$ point B. Since the points are located on the unit circle, the coordinates of the points can be defined as A(cos α , sin α) and B(cos β, sin β). The distance between these two points can then be determined by using the distance formula:

$$AB^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$

$$AB^2 = (\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta) + (\sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta)$$

$$AB^2 = (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$AB^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

Next, let the unit circle, and all of the points on it, be rotated, such that point B is at the coordinates B(1,0). This then makes the coordinates of point A be A(cos s, sin s), for s= α - β. The value of s will be substituted in later. The distance between these two points can be defined as:

$$AB^2 = (\cos s - 1)^2 + (\sin s - 0)^2$$

$$AB^2 = (\cos^2 s - 2\cos s + 1) + \sin^2 s$$

$$AB^2 = (\sin^2 s + \cos^2 s) - 2\cos s + 1$$

$$AB^2 = 2 - 2\cos s$$

$$AB^2 = 2 - 2\cos(\alpha - \beta)$$

Since both of the final statements are equal, they can be made equal to each other:

$$2 - 2\cos(\alpha - \beta) = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$-2\cos(\alpha - \beta) = -2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

This formula can be used to find the difference between two numbers within a cosine function. In order to obtain the law of tangents, though, the sum and difference formulas for sines are needed. Fortunately, the difference formula for cosines can be used to create these two functions. First, recall the co-function identities for sines and cosines:

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \text{ and } \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

These two co-function identities can be used with the difference formula for cosines to create the addition formula for sines:

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right]$$

$$\sin(\alpha + \beta) = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$$

$$\cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right] = \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

To create the difference formula for sines, β was replaced by -β:

$$\sin\left[\alpha + (-\beta)\right] = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Adding and subtracting the sum and difference formulas for sines yields the following:

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$$

In order to continue simplifying the pair of equations, the variables will need to be replaced. $\alpha + \beta$ will be represented by A, and $\alpha - \beta$ will be represented by B. Using some algebraic skills, it can be shown that α will be represented by $\frac{A+B}{2}$ and β will be represented by $\frac{A-B}{2}$. Substituting these into the previous pair of equations yields the following:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Now that these equations are in an easier form to work with, the first equation will be divided by the second. This will produce two proportions, as demonstrated in the following:

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

With this equation, the right side can be simplified by recalling that $\frac{\sin\theta}{\cos\theta} = \tan\theta$. The left side, on the other hand, has to be simplified by using the law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

When the first part of the law of sines is rewritten, it can then be used in the left side of the equation.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\sin A = \frac{a\sin B}{b}$$

Finally, both sides of the equation can be rewritten and further simplified to derive the law of tangents:

$$\frac{\left(\frac{a\sin B}{b}\right)+\sin B}{\left(\frac{a\sin B}{b}\right)-\sin B}=\frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$\frac{a\sin B+b\sin B}{a\sin B-b\sin B}=\frac{\sin\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}\cdot\frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}$$

$$\frac{(a+b)\sin B}{(a-b)\sin B}=\tan\left(\frac{A+B}{2}\right)\cdot\frac{1}{\tan\left(\frac{A-B}{2}\right)}$$

$$\frac{a+b}{a-b}=\frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$