

Vector Calculus Scalar field: $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, Vector field $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

- Line integral of scalar fields (integral with respect to arc length)
 - Notation: $\int_C f ds$
 - Evaluation: $\int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt$ where $\vec{r}(t)$ is a parametric representation of the curve C .
 - Mass of wire: $m = \int_C \rho ds$ where ρ is the linear density of the wire.
 - Line integral of a constant function: $\int_C k ds = k \cdot Length(C)$
- Line integral of vector fields (If the curve is closed the line integral is called **circulation**):
 - Notation: $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds$, \vec{T} is the unit tangent vector to the curve C .
 - Different notation: $\int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$ where $\vec{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$
 - Evaluation: $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$
 - Reversed path: $\int_{C_{rev}} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r}$
 - Application: Work of force field on moving object.
- Conservative vector fields
 - Characterizations:
 - (a) \vec{F} is conservative ($\oint_C \vec{F} \cdot d\vec{r} = 0$ for all simple closed curve C).
 - (b) \vec{F} is a gradient field, has a potential f ($\vec{F} = \nabla f$)
 - (c) $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path, i.e. only depends on the endpoints of C .
 - (d) $\nabla \times \vec{F} = 0$ (if \vec{F} is nice)
 - Fundamental theorem of line integrals: If \vec{F} is a gradient field with potential f ($\vec{F} = \nabla f$) then $\int_C \vec{F} \cdot d\vec{r} = f(P_1) - f(P_0)$ where P_0 and P_1 are the initial and terminal point of the curve C .
- Rotation \equiv curl
 - Notation: $\text{curl } \vec{F} = \nabla \times \vec{F}$
 - Definition in 2D: $\frac{\partial}{\partial x} Q(x, y) - \frac{\partial}{\partial y} P(x, y)$
 - Definition in 3D:
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y, z) & Q(x, y, z) & R(x, y, z) \end{vmatrix}$$
- Divergence
 - Notation: $\text{div } \vec{F} = \nabla \cdot \vec{F}$
 - Definition: $\frac{\partial P(x, y, z)}{\partial x} + \frac{\partial Q(x, y, z)}{\partial y} + \frac{\partial R(x, y, z)}{\partial z}$
 - Interpretation: flux density, source or sink
- Surface integral of scalar field
 - Notation: $\iint_S f dS$
 - Evaluation: If the surface S is parametrized by $\vec{r}(u, v)$ with u and v in the domain D , then: $\iint_S f dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| dA$
 - Applications:
 - Mass of aluminum foil: $m = \iint_S \rho dS$ where ρ is the area density.
 - Surface integral of a constant function: $\iint_S k dS = k \cdot Area(S)$
- Surface (flux) integral of vector fields
 - Notation: $\iint_S \vec{F} \cdot d\vec{S}$
 - Alternative notation: $\iint_S \vec{F} \cdot \vec{n} dS$ where \vec{n} is the normal vector to the oriented surface.
 - Evaluation: $\pm \iint_R \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u(u, v) \times \vec{r}_v(u, v)) dA$ depending on the orientation.
 - Interpretation: stuff going through the surface
- Stokes' theorem
 - Idea: the integral of the curl on a surface is the circulation on the boundary.
 - If $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and C is the boundary of S (S is on the left of C) then $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$
- Green's theorem (2D version of Stoke's theorem)
 - If $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and C is the boundary of R (R is on the left of C) then $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} dA$
- Divergence theorem (Gauss' Theorem)
 - Idea: the integral of flux density is the flux on the boundary, i.e. amount coming in and out through the boundary (flux) is equal to the amount created and swallowed (divergence) inside.
 - If $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and S is the boundary (outward orientation) of the solid E then $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} dE$