Objectives: At the end of this lesson, you should be able to find the strict local maximum and minimum.

Background

The function \( y = f(x) = 0.35x^3 - 3.5x^2 + 9x + 1 \) on the right has easily identifiable local extreme points at each of its turning points. Graphically, there is a local maximum at about \((1.74, 7.91)\) and a local minimum at \((4.93, 2.24)\). From the graph it is also obvious that there is no absolute extrema. The function plunges to the left and zooms to the right.

You can find the local extreme points by finding the first derivative. In this case, \( y' = 1.05x^2 - 7x + 9 \). Setting \( y' = 0 \) to solve for \( x \), will eventually lead you to the quadratic formula or a graph since it doesn’t factor.

Of course, we have the second derivative test to show whether any critical value where \( y' = 0 \) is an extreme value when it isn’t as obvious as this case. You should recall that when \( y' = 0 \) and \( y'' > 0 \) you’ll find a minimum. If you don’t, use my “memory trick.” You know that \( y = x^2 \) opens up and has a minimum. It doesn’t take but a second to find \( y'' = 2 > 0 \). If that doesn’t work, remember a happy face has both a positive attitude and second derivative.

Finding Local Extreme Points in Space

Finding local extreme points in three dimensions follow the same idea. Here, you equate the two first-order partials to 0 and solve them (simultaneously in many cases).

Look at these examples of local extreme points. The first graph has a local maximum. The second one has a local minimum. The third and fourth have a local maximum in the direction of one of the axes and a local minimum in the other, i.e. it has neither a local maximum nor a local minimum, and is called a saddle point. However the fourth also has very clear local maximums (the camel humps) with a saddle point.
Local Extreme Points

Theorem – Second Derivative Test for Local Extrema

Let \( f(x, y) \) be a function with continuous second order partials in the domain \( S \), and let \( (x_0, y_0) \) be an interior point of \( S \) that is a stationary point for \( f \).

1. If \( f_{xx}(x_0, y_0) < 0 \) and \( f_{xx}(x_0, y_0)f_{xy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0 \), then \( (x_0, y_0) \) is a (strict) local maximum point.

2. If \( f_{xx}(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0)f_{xy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0 \), then \( (x_0, y_0) \) is a (strict) local minimum point.

3. If \( f_{xx}(x_0, y_0)f_{xy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 < 0 \), then \( (x_0, y_0) \) is a saddle point.

4. If \( f_{xx}(x_0, y_0)f_{xy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 = 0 \), then \( (x_0, y_0) \) could be a local maximum, a local minimum, or a saddle point.

You’ll be pleased to know that there is no new mathematical process in this section. We are just employing the partial derivatives for a specific purpose.

In general, the more beautiful a graph (surface) is the more complex is its equation. Don’t let this bother you. Most graphs used in business are relatively simple. Hence, they aren’t really all that beautiful.

Example

Suppose \( z = f(x, y) = x^2 + xy + y^2 + x - 3y + 7 \). Find the local extremes. State the location and whether a maximum or a minimum.

We find the partials first.

\[
\begin{align*}
  f_x &= 2x + y + 1 = 0 \text{ implies that } y = -2x - 1 \\
  f_y &= x + 2y - 3 = 0 \text{ implies that } x = -2y + 3 \\
  f_{xx} &= 2 > 0, \quad f_{yy} = 2 > 0, \quad f_{xy} = 1 \text{ are the second order partials and do not depend on } (x_0, y_0)
\end{align*}
\]

Notice that \( f_{xx} = 2 > 0 \). A local maximum is not possible. So if our mission were to find only a maximum, we can stop!

Now, solving the two first order partials simultaneously by substitution into \( x = -2y + 3 \).

We get \( x = -2(-2x-1)+3 = 4x + 5 \). Then \( -3x = 5 \) and \( x = -5/3 \).

So, by back substitution, \( y = -2x - 1 = -2(-5/3) - 1 = 7/3 \).

Our critical point \( (x_0, y_0) = (-5/3, 7/3) \).

Calculating \( f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 = 2 \cdot 2 - 1^2 = 3 > 0 \), so we have a local minimum.

Now find the minimum value of the function by substituting \((-5/3, 7/3)\) into \( f(x, y) \).

\[
  f(-5/3, 7/3) = (-5/3)^2 + (-5/3)(7/3) + (7/3)^2 + (-5/3) - 3(7/3) + 7 = 8/3
\]

Thus, there is a local minimum of \( 8/3 \) at \((-5/3, 7/3)\).