The Lagrange Multiplier Method

Objective: At the end of this lesson, you should be able to find the absolute maximum and minimum using the direct substitution method and the Lagrangian method.

Background

So far, we have optimized a function subject to some constraint using the direct substitution method. This works well as long as the constraint can be solved for one of the variables involved.

Example

1. Find the maximum and minimum values of \( f(x, y) = 4xy + 5 \) under the constraint
\[
2x^2 + 3xy + 5y^2 = 10
\]

It would be difficult to solve this problem using the direct substitution method. However, the Lagrange method is much easier. This will be shown later in this lecture.

2. Find the minimum value of \( f(x, y) = 3x^2 + 4y^2 \) under the constraint \( 4x + 5y = 20 \).

Now this constraint can be easily solved for either \( x \) or \( y \) and the direct substitution method works fine.

Solve the constraint \( 4x + 5y = 20 \) to get \( y = 4 - \frac{4x}{5} \). Then substitute. The result is to the right. Then set
\[
 f'(x) = \frac{278}{25}x - \frac{128}{5} = 0
\]

Back substitute to find \( y = 4 - \frac{4(2.302)}{5} = 2.158 \).

Evaluate to find the minimum value:
\[
f(2.302, 2.158) = 3(2.302)^2 + 4(2.158)^2 = 34.525.
\]

State that the minimum value of the function is 34.525 at the point \((2.302, 2.158)\). Finish the problem!

The Lagrange Theorem

Suppose you are given the function \( f(x, y) \) subject to the constraining function \( g(x, y) = c \). Suppose both functions have continuous partial derivatives on a domain \( A \) in the \( xy \)-plane. Suppose that the point \((x_0, y_0)\) is an interior extreme point the domain under the constraint \( g(x, y) = c \) and \( g_x(x_0, y_0) \neq 0 \) and \( g_y(x_0, y_0) \neq 0 \) (one them could be zero). Then there is a unique constant \( \lambda \) (lambda to honor the L in Lagrange) where the function
\[
\ell(x, y) = f(x, y) - \lambda (g(x, y) - c)
\]
has a stationary point at \((x_0, y_0)\).

As weak as it sounds, the authors of your textbook have done as good a job as anyone in validating this result. Let’s focus on applying it.
Using The Lagrange Multiplier Method

Given a \( f(x, y) \) subject to \( g(x, y) = c \), find the absolute maximum (or minimum) on the domain through this process.

1. Construct the Lagrangian function \( \ell(x, y) = f(x, y) - \lambda (g(x, y) - c) \) for some constant \( \lambda \). The constant \( \lambda \) is the multiplier.
2. Find the two first partials \( \ell_x(x, y) \) and \( \ell_y(x, y) \).
3. Solve the system \( \ell_x(x, y) = 0 \), \( \ell_y(x, y) = 0 \), and \( g(x, y) = c \) simultaneously for \( x, y, \) and \( \lambda \).

Please note that the systems you will solve at this point should all work through simple substitutions. Later in the course we will learn to solve much more difficult systems. For now, focus on the calculus issues.

Examples

1. **Maximize** \( f(x, y) = x^2 + 4xy + y^2 \) subject to \( x + y = 100 \).

   **Step 1:** Build the Lagrangian function: \( \ell(x, y) = x^2 + 4xy + y^2 - \lambda (x + y - 100) \)

   It is probably best not to simplify in any sense. The partials will be easier most of the time.

   **Step 2:** Find the partials: \( \ell_x(x, y) = 2x + 4y - \lambda = 0 \) and \( \ell_y(x, y) = 4x + 2y - \lambda = 0 \)

   \[
   \begin{cases}
   2x + 4y - \lambda = 0 \\
   4x + 2y - \lambda = 0
   \end{cases}
   \]

   **Step 3:** Solve the system \( 2x + 4y = \lambda \) and \( 4x + 2y = \lambda \) for all symbols.

   Subtracting the second from the first gives \( -2x + 2y = 0 \) will eliminate the \( \lambda \) and \( y = x \).

   Substituting \( y = x \) into the third equation \( x + y = 100 \) gives \( 2x = 100 \), \( 2x = 100 \) and \( x = 50 \)

   So, \( y = x = 50 \) and the function is maximized at the point \((50,50)\)

   **Step 4:** State the solution! Since \( f(50,50) = 50^2 + 4(50)(50) + 50^2 = 15000 \), the maximum value of the function is 15,000 at \((x,y) = (50,50)\).

Take a look at the function \( f(x, y) = x^2 + 4xy + y^2 \). It is symmetric \textit{wrt} \( x \) and \( y \). It shouldn’t be a surprise that the solution point has the form \((k,k,f(k,k))\).
The Lagrange Multiplier Method

2. Find two positive numbers whose sum is 40 such that the sum of their squares is as small as possible.

Finding these two numbers will result from minimizing \( f(x, y) = x^2 + y^2 \) subject to \( x + y = 40 \)

This can be solved easy by either methods. We will use the Lagrange multiplier method. The steps are shown without explanation. You will need to show this same detail!

\[
\ell(x, y) = x^2 + y^2 - \lambda(x + y - 40)
\]

\[
\ell_x(x, y) = 2x - \lambda = 0
\]

\[
\ell_y(x, y) = 2y - \lambda = 0
\]

Solve: \[
\begin{aligned}
2x - \lambda &= 0 \\
2y - \lambda &= 0 \\
x + y &= 40
\end{aligned}
\]

So, \( y = x = 20 \) and the numbers 20 and 20 will sum to 40 and the sum of their squares will be smallest.

If you think about this problem, you may remember seeing it in Brief Calculus and/or in algebra.

The example above would have been very difficult using the direct substitution method but can be solved using the Lagrange multiplier method fairly easily.

3. Find the maximum and minimum values of \( f(x, y) = 4xy + 5 \) under the constraint

\( 2x^2 + 3xy + 5y^2 = 10 \).

\[
\ell(x, y) = 4xy + 5 - \lambda \left( 2x^2 + 3xy + 5y^2 - 10 \right)
\]

\[
\ell_x(x, y) = 4y - \lambda \left( 4x + 3y \right) = 0
\]

\[
\ell_y(x, y) = 4x - \lambda \left( 3x + 10y \right) = 0
\]

Solve: \[
\begin{aligned}
4y - \lambda \left( 4x + 3y \right) &= 0 \\
4x - \lambda \left( 3x + 10y \right) &= 0 \\
2x^2 + 3xy + 5y^2 &= 10
\end{aligned}
\]

Solve each of the first two equations for \( \lambda \): \[
\frac{4y}{4x + 3y} = \lambda, \quad \frac{4x}{3x + 10y} = \lambda \]. So, \( \frac{4y}{4x + 3y} = \frac{4x}{3x + 10y} \).

Cross multiply: \( 4y(3x + 10y) = 4x(4x + 3y) \)

Simplify: \( 12xy + 40y^2 = 16x^2 + 12xy \) \( \Rightarrow 40y^2 = 16x^2 \) \( \Rightarrow y^2 = \frac{16}{40}x^2 = \frac{2}{5}x^2 \) \( \Rightarrow y = \pm \sqrt{\frac{2}{5}}x \)

Substitute \( y = \sqrt{\frac{2}{5}}x \) and \( y = -\sqrt{\frac{2}{5}}x \) one at a time into the third equation \( 2x^2 + 3xy + 5y^2 = 10 \).
Then solve for $x$. Once you get the simplification after the substitution, you can also use the quadratic formula.

For $y = \sqrt[5]{2} x$:

$$2x^2 + 3xy + 5y^2 = 10$$

$$2x^2 + 3\sqrt[5]{2}x + 5\frac{2}{5}x^2 = 10$$

$$4x^2 + 3\sqrt[5]{2}x^2 = 10$$

multiply by $\sqrt[5]{5}$

$$4\sqrt[5]{5}x^2 + 3\sqrt[2]{}x^2 = 1$$

$$\left(4\sqrt[5]{5} + 3\sqrt[2]{}\right)x^2 = 1$$

factored

$$x^2 = \frac{10\sqrt[5]{5}}{4\sqrt[5]{5} + 3\sqrt[2]}$$

division

$$x = \pm \sqrt[5]{\frac{10\sqrt[5]{5}}{4\sqrt[5]{5} + 3\sqrt[2]}} \approx \pm 1.3022$$

square root

$$y = \pm \sqrt[5]{\frac{2}{5}x} = \pm \sqrt[5]{\frac{2}{5}(1.3022)} = \pm 0.8236$$

back substitution

This gives us four points:

$$(1.3022, 0.8236), (1.3022, -0.8236), (-1.3022, 0.8236), (-1.3022, -0.8236).$$

When $y = -\sqrt[5]{2}x$, a similar process leads to $x \approx \pm 10.2534, y \approx \pm 6.4848$ and four more points:

$$(10.2534, 6.4848), (10.2534, -6.4848), (-10.2534, 6.4848), (-10.2534, -6.4848)$$

Now, evaluate all of them to find the (approximate) extremes. The table summarizes the result. The first calculation is shown.

When $(1.3022, 0.8236)$, $f(x, y) = 4xy + 5 = 4(1.3022)(0.8236) + 5 = 9.2900$

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$(1.3022, 0.8236)$</th>
<th>$(1.3022, -0.8236)$</th>
<th>$(-1.3022, 0.8236)$</th>
<th>$(-1.3022, -0.8236)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x, y)$</td>
<td>9.290</td>
<td>0.7100</td>
<td>0.7100</td>
<td>9.290</td>
</tr>
<tr>
<td>$(x, y)$</td>
<td>(10.2534, 6.4848)</td>
<td>(10.2534, -6.4848)</td>
<td>(-10.2534, 6.4848)</td>
<td>(-10.2534, -6.4848)</td>
</tr>
<tr>
<td>$f(x, y)$</td>
<td>270.9650</td>
<td>-260.9650</td>
<td>-260.9650</td>
<td>270.9650</td>
</tr>
</tbody>
</table>

The maximum of $f(x, y)$ is 270.9650 at the points $(10.2534, 6.4848)$ and $(-10.2534, -6.4848)$

The minimum of $f(x, y)$ is -265.9650 at the points $(10.2534, -6.4848)$ and $(-10.2534, 6.4848)$