Matrix Operations

Objectives: At the end of this lesson, you should be able to:

1. Define the matrix operations.
2. Apply the matrix operations.
3. Create a matrix equation.

Background

Matrices create a mathematical structure that has many properties similar to the real numbers system. They also have some distinctly different properties. The differences are where the fun comes in.

Think back on the real number system. We can add, subtract, multiply and apply powers and apply a multiplicative inverse to create results the real number system.

We can do the same thing with matrices as long as we obey some simple rules. Most of the rules require us to pay attention to the size of the matrix.

Addition and Subtraction

We can add or subtract matrices of exactly the same size. We do this by combining the elements within the matrix in the same (corresponding) position.

Let’s name our matrices $A$ and $B$ where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$.

Then $A \pm B = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$.

Notice that this requires the matrices to both be $m \times n$. If they are not, the sum is not defined.

Examples:

$\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 10 & 0 \\ 0 & 11 \end{pmatrix} = \begin{pmatrix} -1 + 10 & 3 + 0 \\ 2 + 0 & 1 + 11 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 2 & 12 \end{pmatrix}$

The two matrices are both $2 \times 2$, so we add element by element.

$\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 10 & 4 & 0 \\ 0 & 5 & 11 \end{pmatrix}$

These two matrices are different sizes ($2 \times 2$ and $2 \times 3$). We cannot add them. This is undefined.

$\begin{pmatrix} -1 & 3 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 10 & 4 & 0 \\ 0 & 5 & 11 \end{pmatrix}$

These two matrices are different sizes ($3 \times 2$ and $2 \times 3$). We cannot add them. This is also undefined.
Matrix Operations

The two matrices are interesting! They create a field of zeroes in the resulting $2 \times 2$ matrix. The two matrices are additive inverses. Every matrix $A$ has an additive inverse, called $-A$. It is created by reversing the sign of every entry in $A$.

The additive inverses in the example also creates the zero matrix for the $2 \times 2$ set through addition. The zero matrix has only zeroes in each position. Each size of matrices has its own zero matrix. It is sometimes represented by a big, bold zero, $0$.

This is also called the additive identity since adding or subtracting it from any matrix $A$ of the same size, leaves the matrix unchanged: $A \pm 0 = A$.

$$
\begin{pmatrix}
1 & 2 \\
3 & 1
\end{pmatrix}
+ \begin{pmatrix}
-1 & -2 \\
-3 & -1
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
$$

Even though it seems like you could write this as $A + 0 = A$, the zero matrix is from a different-sized set. This is an apples-and-oranges situation and is still undefined.

Matrix Multiplication

We can multiply matrices when they have a very specific size relationship. Let’s use about the simplest example of matrix multiplication to show how it works.

Let $A = \begin{pmatrix} a & b \end{pmatrix}$. We call this a row matrix since it has only one row.

Let $B = \begin{pmatrix} c \\ d \end{pmatrix}$. Recall that this is a column matrix.

The multiplication process is a row by column process. We can multiply $AB = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = (ac + db)$.

Notice we have a $(1 \times 2)$ times a $(2 \times 1)$ to create a $(1 \times 1)$ matrix. This is called an inner product (or dot product). The inner sizes (2 in each case) must match. The resulting size of the product is the same as the two outer sizes (1 in each case).

If we try to do $BA$, we have a $(2 \times 1)$ times $(1 \times 2)$. It still works, but we create a $(2 \times 2)$ matrix!

$$
BA = \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} ca & cb \\ da & db \end{pmatrix}
$$

**When we multiply two matrices, the product may be defined in one direction and not in the other.**

Let $C = \begin{pmatrix} c & e \\ d & f \end{pmatrix}$. Notice that $AC = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} c & e \\ d & f \end{pmatrix} = (ac + bd \quad ae + bf)$. However, $CA$ is not defined at all.

Similarly, $CB = \begin{pmatrix} c & e \\ d & f \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c^2 + ed \\ dc + fd \end{pmatrix}$, but $BC$ is not defined.
Matrix Operations

You probably wonder why we even care to multiply matrices in any direction. That’s a fair thought. Let’s create a couple of examples where we might do such a multiplication.

**Creating a Matrix Equation**

**Example:** A company has three machines called \( x_1, x_2 \) and \( x_3 \) which produce the same part. These machines require 3 hours, 4 hours and 5 hours in maintenance daily.

We can create the row vector of maintenance needs as \( \begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \) and the column vector of variables representing the number of each type of machine as \( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \).

Then the total hours of maintenance is the product \( \begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \).

Now this might not seem very much of a savings of writing effort. So let’s add some more information about these machines.

These machines can produce 30, 60, and 150 of the same kind of part each in a day.

That is reflected as the product \( \begin{pmatrix} 30 & 60 & 150 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \).

Still seems like a lot of work until we realize we can combine the two statements in a single product.

\[
\begin{pmatrix} 3 & 4 & 5 \\ 30 & 60 & 150 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

Now when I tell you that the cost of the machines are $125K; $200K and $350K dollars respectively, you realize that all of this information can be summarized in the product

\[
\begin{pmatrix} 3 & 4 & 5 \\ 30 & 60 & 150 \\ 125 & 200 & 350 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

That is definitely using the product to create an effective summary. If we just stopped there I could be happy with what I’ve learned. But there is so much more we can do.

**Recognize that this is a (3×3)(3×1) to create a (3×1) matrix. What would that matrix product describe?**

Reading back the to first line set-up, you may have caught the word “total.” The product is the totals of maintenance hours, production capability and cost of procurement for that same number of machines.

The finally surprise piece of data is that the company in question schedules 120 daily maintenance hours, needs a production level of 2400 items, and has a budget of $6.75 million dollars to buy the machinery.
Matrix Operations

This creates the \((3\times1)\) column matrix of constants \[
\begin{pmatrix}
120 \\
2400 \\
6750 
\end{pmatrix}
\] which reflects these totals.

Now we can complete the package

\[
\begin{pmatrix}
3 & 4 & 5 \\
30 & 60 & 150 \\
125 & 200 & 350
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
120 \\
2400 \\
6750
\end{pmatrix}
\]

This matrix form is equivalent to the system of equations to the right.

I hope I didn’t slip one by you when I scaled the budget line in thousands. Until I created the final product, I really was playing I’ve got a secret there.

However, I dropped little pieces of information to show the thinking as I built the product. The problem would more likely appear this way:

Write the matrix equation modeling the following information:

A company has three machines called \(x_1\), \(x_2\) and \(x_3\) which produce the same part. These machines require 3 hours, 4 hours and 5 hours in maintenance daily. The company allocates 120 hours to maintenance daily. These machines can produce 30, 60, and 150 of the same kind of part each in a day. The company needs to produce (and sell) 2400 parts daily to meet its contracts. The cost of the machines are $125K; $200K and $350K dollars respectively. Finally, the company budgets $6.75 million to buy the machines.

Notice the words “matrix equation” in the instructions.

Be careful to respond to the question properly. It is true that both of the forms boxed to the right reflect the same model when placed in context. However, each has a proper title. Please use them! Vocabulary does matter.