Matrix Inverse

Objectives: At the end of this lesson, you should be able to:

1. Understand why we want to find a matrix inverse.
2. Understand why we cannot always find a matrix inverse.
3. Apply Gaussian Elimination to find a matrix inverse

Background

A critical piece in the use of matrices is the inverse. Let’s look back at how we used the multiplicative inverse to solve simple algebraic equations.

Suppose you wanted to solve $4x = 8$. You would probably think “divide by 4”. However, notice that we never defined a division process for matrices. Thinking back to your beginning algebra, you recall that you were told to apply the multiplicative inverse (usually called the reciprocal in arithmetic).

The fancy way to do this is to write the reciprocal as $4^{-1}$. Then we get $4^{-1}4x = 1x = 4^{-1}8$. After all that $x = 2$. We can solve matrix equations exactly the same way.

The Inverse Matrix Defined

Suppose someone told you to write the solution for $AX = B$ where $A$, $X$, and $B$ are matrices. IF (big if) we knew that there is a matrix out there that is the inverse of $A$, the solution would be $A^{-1}AX = IX = A^{-1}B$.

So let’s begin with a definition:

For the $n \times n$ matrix $A$, if an $n \times n$ matrix $B$ exists so that $AB = BA = I_n$, we call it the inverse of $A$. We use the symbol $A^{-1}$ to represent the inverse of $A$. We will tell you that $B$ is unique. For each matrix $A$ with an inverse, there is only one $A^{-1}$.

There is a ton of stuff going on in this definition. The only way we can guarantee both a left and right side product is that the matrix $A$ must be square ($n \times n$). So the product always creates the identity matrix of the same size.

That’s where it stops being simple. In the number world, we can write the reciprocal as “one over the number” then cancel, divide, multiply, ... whatever works best. Even if we did, this it would not tell us how to write the inverse matrix in a practical way.

Example: Look at $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. What would $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ really mean anyway? I promise you $A^{-1}$ is not $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}$.

Try the product of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ to prove it to yourself.
How to Find the Inverse

So how do we find the inverse matrix? *Gaussian elimination* is the mechanism. We use a special matrix form \((A : I_n)\) where the matrix of interest \(A\) is augmented with the appropriate identity matrix \(I_n\).

Next we row reduce until we achieve \((I_n : A^{-1})\) or prove that we cannot find an inverse. Remember, some matrices are bad actors. They act like zero for things like multiplication and inverses. *Even in the matrix world, the zero matrix cannot have an inverse.*

**Example:** Find the inverse for \(A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\).

We create the matrix form \((A : I_2) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix}\). Then row reduce using these steps: 

\[
R_1 = r_1 - r_2, \\
R_2 = \frac{1}{2} r_2
\]

You can verify that the steps do create \((I_2 : A^{-1}) = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -\frac{1}{2} \end{pmatrix}\).

Now we know that \(A\) has an inverse and it is \(A^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & -\frac{1}{2} \end{pmatrix}\).

**Example:** Find the inverse for \(A = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 4 & 6 \\ 0 & 1 & 2 \end{pmatrix}\).

We create the matrix form \((A : I_3) = \begin{pmatrix} 1 & 2 & 5 & 1 & 0 & 0 \\ 1 & 4 & 6 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}\). Then row reduce using these steps: 

\[
R_3 = r_3 - r_1, \\
R_2 = 2r_2 - r_3, \\
R_3 = \frac{1}{3} r_3
\]

This creates this matrix form: \(\begin{pmatrix} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\).

Continue row reducing (you finish it) until you have \((I_3 : A^{-1}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\).

Again we know that \(A\) has an inverse and \(A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\).
Matrix Inverse

Example: Find the inverse for \( A = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 5 & 7 \end{pmatrix} \) or show that \( A^{-1} \) does not exist.

Create \( (A | I_3) = \begin{pmatrix} 2 & 4 & 6 & 1 & 0 & 0 \\ 3 & 6 & 9 & 1 & 0 & 0 \\ 4 & 5 & 7 & 0 & 0 & 1 \end{pmatrix} \).

Now row reduce. If \( A^{-1} \) doesn’t exist, we should arrive at a row with all zeroes on the \( I_n \) side.

This one is simple. Using the linear combination \( R_2 = 3r_2 - 2r_1 \) zeroes out the second row where we are supposed to create the \( I_3 \) form. That does it. There is no way to create the \( (I_3 : A^{-1}) \) form. No inverse!

Before you accuse me of misleading you about the “simplicity” of using an inverse, I do have good news. The calculator parked next to you has the capability to find an inverse if it exist. However, if the inverse does not exist, your would need to row reduce to find the general solution of a system or know that there is no solution.

In a business situation, the inverse probably does exist. If not, either your model is incorrect from the beginning, or you’re in the wonderful situation where you have infinitely many ways to run the business! Let’s do two more examples.

Example: Find the inverse matrix for \( A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \).

I do admit that this is a really lazy one to do! However, the method is just as easily shown here as with any matrix.

Start by creating the \( (A : I_4) \) augmented form. \( A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \)

From there you should see that we just need to use the combination \( R_3 = (2r_1 + 3r_2) - r_3 \).

This give the result of \( A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \).
**Matrix Inverse**

**Example:** Find the inverse matrix for \( A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \).

As much as this looks like the last example, look closely at the third row of the matrix.

Start by creating the \( \begin{pmatrix} A : I_4 \end{pmatrix} \) augmented form.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

We still need to use the combination \( R_3 = (2r_1 + 3r_2) - r_3 \). However, this time we are doomed!

It gives the result of

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
2 & 3 & 0 & 0 & -2 & -3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

A little observation would have supported the thought that either there are only three variables or one of them is “freebie”. In any case no inverse exists.

For more difficult situations we will depend on our calculator. If you check your on-line lessons, you will find a video about how to use the TI-83/84 calculator. The written explanation is truly tedious for such a simple process.