Background

The core of all probability calculations is the Outcome Space, usually indicated with $S$. This is the complete listing of all possible outcomes. Unlike its set counterpart, the universal set, a Outcome Space is usually put in order to help our analyses. Also, if it is useful, we can list an element as often as it might occur.

Example: We have a bag with five balls in it. Two are numbered 1, two are numbered 2, and one is numbered 3. Our game is to randomly draw a ball from the bag one time.

The Outcome Space is the set $S = \{1, 2, 3\}$. However, we could equally well record the Outcome Space as $S = \{1,1,2,2,3\}$. Both are complete listings of the possible outcomes.

It is easy to see that we have $P(1) = \frac{2}{5}, P(2) = \frac{2}{5}, P(3) = \frac{1}{5}$. Notice that the fractions add to one.

So $P(S) = 1 = 100\%$. This must always check.

Let’s expand the concept and see where it might be useful to use the shorter listing of outcomes, then through some mechanism indicate the frequency ($f$) of each outcome.

We have a new bag with five hundred balls in it. Two hundred are numbered 1, two hundred are numbered 2, and one hundred are numbered 3. The Outcome Space is still the set $S = \{1, 2, 3\}$, but none of us wants to write down the list of five hundred choices!

Probability Model Review

A probability model summarizes the results obtained from a process whether by counting, a tree diagram, or possibly even a Venn diagram. Characteristics of a valid model are

1. All outcomes are listed from the Outcome Space.
2. No impossible events are listed.
3. All probabilities are listed for the outcomes. Also, 
   a. The total of all probabilities is one $= 100\%$.
   b. All probabilities are strictly between zero and one.

Methods of showing a probability model include the following:

- Summarize the outcomes into categories when useful.
- Develop a frequency table when useful. Frequency tables record an outcome with the number of times it occurs.
- Calculating relative frequency ($rf$), the ratio of the number of times an outcome occurs to the number of trials. This is often recorded as a decimal value, but use caution. If you need to round, the total may not equal one!
Back to the two bags above.

The probability model for each of them is below:

<table>
<thead>
<tr>
<th>Bag with Five Balls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outcome</td>
</tr>
<tr>
<td>frequency (f)</td>
</tr>
<tr>
<td>(P(\text{Outcome}) = rf(\text{Outcome}))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bag with Five Hundred Balls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outcome</td>
</tr>
<tr>
<td>frequency (f)</td>
</tr>
<tr>
<td>(P(\text{Outcome}) = rf(\text{Outcome}))</td>
</tr>
</tbody>
</table>

Now I know you will not convince most 7-year-old kids, but ... If the balls were numbered jaw breakers, and the ones with the number 1 on them were their favorite flavor, they have exactly the same chance of success in randomly picking their favorite from the bag with five balls as the bag with five hundred! Notice that the relative frequencies for ball numbers are the same in the tables when we reduce them. These are exactly the same models for a one draw situation.

More Examples

_Do the following_: For each of the following situations list the Outcome Space then build a probability model for the process described. From the model, calculate or state the probabilities for the events described.

1. You flip a fair a coin then choose a number from the bag containing only \(\{1, 2, 3, 5\}\).
   
   What is the probability of getting a heads followed by the number 3?
   What is the probability of getting a tails followed by the number 4?
   What is the probability of getting a heads followed by a number smaller than 4?

2. Choose a number from the bag containing only \(\{1, 4, 5, 6\}\), replace the number, then choose another number from the same bag.
   
   What is the probability of getting the number 4?
   What is the probability of getting a 4 followed by the number 5?
   What is the probability of getting a total of 9?
   What is the probability of getting a total of 10 when you add the digits?

   Check your results by clicking in the checkmark before you go on.

Read the next example carefully. The outcome space starts the problem, but applying correct probabilities is an equally critical step. It also demonstrates how we work with experimental, empirical, or accumulated data.
Outcome Spaces In Probability

Example 3: Professor Turner is notorious for being late to class. (Hah! A vicious canard!) The students have observed that he is late 42% of the time. Unfortunately they have noticed that when he is on time, they get a pop quiz about 85% of the time. When he is late, they get a pop quiz about 30% of the time. What is the probability he gives a pop quiz on any given class day?

Before we tackle Example 3, let’s talk about how this empirical data about tardiness and quizzes works for us.

The Law of Large Numbers

If an experiment or trial is repeated a large number of times, the relative frequency of an outcome is approximately the same as the theoretical probability for that outcome. What this means in probability:

1. We can use empirical results taken from a large experiment as a reasonable assignment of probabilities for an Outcome Space.
2. We can use theoretical probability as an estimator for the likelihood of an event really happening if we repeat the action creating the event a large number of times.

Now back to Example 3. In this case, we assume that the Mr. T’s track record is well enough documented (Hah again!) that the percentages provided can be used as probability statements.

The best way to start a problem like this is through a diagram. Since we can structure it this way, the tree will be small.

Step 1: Branch to Late or not late

Step 2: Based on late or not late, branch to either quiz or no quiz.

We can record the outcomes right now. There are four of them:

\[ S = \{ \text{late} \rightarrow \text{quiz}, \text{late} \rightarrow \text{no quiz}, \text{not late} \rightarrow \text{quiz}, \text{not late} \rightarrow \text{no quiz} \} \]

Read the little arrow as “and” or as “followed by.” They indicate a directionality applies. This really is my own convention.

This is not a uniform situation. As we will see, none of the probabilities for the outcome space above is 1 in 4.

Look at the tree diagram. We have labeled the probabilities for each step. Notice that at any step, the probabilities assigned to each branch in that option must add to 100% = 1.

How do you think we calculate the probabilities for each outcome?

We can rationalize this by thinking “There is a bag with 42 slips marked late and 58 marked not late in a bag. Similarly there are 30 slips marked quiz and 70 marked no quiz in a second bag. But we only get to choose from this bag when the professor is late. Otherwise we must choose from another set of bags where there are 85 slips marked quiz (bummer) and 15 marked no quiz.”

We can “count” with this scenario. The number of ways to draw from the two bags is 100 × 100. The number of ways to draw late followed by quiz is 42 × 30. So the probability of late → quiz is shown below.

\[
P(\text{late} \rightarrow \text{quiz}) = \frac{42 \times 30}{100 \times 100} = \frac{42}{100} \times \frac{30}{100} = P(\text{late}) \times P(\text{quiz})
\]

1 I have to clarify this also. Professor Turner does not believe in or give pop quizzes. They are a strategy used to motivate children in elementary school. If a college student is not already sufficiently motivated by a $6,000 tuition and desire to have a reasonably decent post-collegiate lifestyle, what good does the “pop quiz” do?
Outcome Spaces In Probability

This is not coincidental. With only a little more effort, we can prove that the next result holds any time we have a sequence of events where counting the first does not affect counting the next. This is called independence.

The Multiplication Principle in Probability

For two sequential events, call them $A$ and $B$, $P(A \text{ and } B) = P(A) \times P(B)$ so long as event $A$ cannot affect event $B$’s probability.

Now we can build the probability model for example 3. Look to the right.

Now it is simple to answer the question, “What is the probability he gives a pop quiz on any given class day?”

We just need to find all the places where quiz is included and add them together. So,

$$P(\text{quiz}) = P(\text{late } \rightarrow \text{ quiz}) + P(\text{not late } \rightarrow \text{ quiz})$$

$$P(\text{quiz}) = 0.126 + 0.493 = 0.619$$

Sadly, there is a 61.9% chance that Mr. Turner gives a quiz on any class day, late or not!

### More Examples of Outcome Spaces and Probability

**Example 4:** How many outcomes are in the outcome space created by listing all possible subsets of three objects drawn from a list of 10 distinct objects?

From the description, this outcome space consists of subsets. Since a set is an unordered list of distinct element, this is a combination process.

The answer is just $C(10, 3)$.

**Example 5:** How many outcomes are in the outcome space created by listing all possible strings of three letters drawn from a list of 10 distinct letters of the alphabet without repeating any letter?

From the description, this outcome space consists of strings. They could also be called words (real or imagined) or codes. Since an outcome is ordered, this is a permutation process.

The answer is $P(10, 3)$.

**Example 6:** How many outcomes are in the outcome space created by listing all possible strings of three letters drawn from a list of 10 distinct letters of the alphabet where repeating letters is allowed?

From the description, this outcome space consists of strings. They could also be called words (real or imagined) or codes.

These outcomes are ordered, but this is not a permutation or combination process because repetition is not allowed within either of them.

The answer is $10^3$. 

---

### Outcome and p(Outcome)

<table>
<thead>
<tr>
<th>Outcome</th>
<th>p(Outcome)</th>
</tr>
</thead>
<tbody>
<tr>
<td>late → quiz</td>
<td>0.126 (0.42 × 0.30)</td>
</tr>
<tr>
<td>late → no quiz</td>
<td>0.294 (0.42 × 0.70)</td>
</tr>
<tr>
<td>not late → quiz</td>
<td>0.493 (0.58 × 0.85)</td>
</tr>
<tr>
<td>not late → no quiz</td>
<td>0.087 (0.58 × 0.15)</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>1.000</td>
</tr>
</tbody>
</table>
Outcome Spaces In Probability

Example 7a: Create a sample space where we draw three letters first from the letters \{A, B, C, D, E, F\} where order matters, then draw four letters from the letters \{G, H, I, K, L, M, N, O\} without concern for order. How large is this space?

Trying to tree this is close to ridiculous. Fortunately, we need a count not a list. We treat the two draws as separate processes to begin with

Step 1: “draw three letters first from the letters \{A, B, C, D, E, F\} where order matters”

This is just \(P(6,3)\).

Step 2: “draw four letters from the letters \{G, H, I, K, L, M, N, O\} without concern for order”

This is just \(C(8, 4)\)

Step 3: This is the critical thinking step. You must have step 1 and step 2 to have a complete process. The rule of thumb is that “and” requires multiplication principle.

The final result is \(P(6,3) \times C(8, 4)\). A typical element of this sample space has an ordered \(n\)-tuple with 7 entries.

Let’s change the wording modestly to see how the result changes.

Example 7b: Create a sample space where we draw three letters first from the letters \{A, B, C, D, E, F\} where order matters, or draw four letters from the letters \{G, H, I, K, L, M, N, O\} without concern for order. How large is this space?

We still treat the two draws as separate processes to begin with

Step 1: “draw three letters first from the letters \{A, B, C, D, E, F\} where order matters”

This is just \(P(6,3)\).

Step 2: “draw four letters from the letters \{G, H, I, K, L, M, N, O\} without concern for order”

This is just \(C(8, 4)\)

Step 3: This is the critical thinking step. You must have step 1 or step 2 to have a complete process. The rule of thumb is that “or” requires addition (or potentially subtraction).

The final result is \(P(6,3) + C(8, 4)\). A typical element of this sample space has an ordered \(n\)-tuple with 3 or 4 entries.

Example 8: A bag has 23 red balls, 42 black balls and 35 green balls.

1. Create a probability model using relative frequencies for a single draw from the bag.

<table>
<thead>
<tr>
<th>Color</th>
<th>Red</th>
<th>Black</th>
<th>Green</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>(\frac{23}{100})</td>
<td>(\frac{42}{100})</td>
<td>(\frac{35}{100})</td>
<td>(\frac{100}{100})</td>
</tr>
</tbody>
</table>

2. Create a probability tree diagram using relative frequencies for a single draw from the same bag in part 1.

The result is the right. Notice that all probabilities assigned to branches add to 1. The same rules apply here as for the tabular probability model.
3. Create a probability tree diagram using relative frequencies for "red" or "not red" followed by a second draw with replacement from the same set of balls.

Look again to the right. Since this is a with replacement process (reusing the ball selected), each of the second step branches is a clone of the first step.

Again, this reflects independence.

4. Create a probability tree diagram using relative frequencies for "red" or "not red" "red" or "not red" followed by a second draw without replacement from the same set of balls.

Really look at the second step branches. In the upper (red chosen) branch, we can put in the fine detail about black and green. Since neither were chosen, we still know how many we have of each.

However, in the lower branch, since red was not chosen we are not able to say for sure how many of the other two colors we have except as a total.

This situation reflects dependence.